Grade 11/12 Math Circles

March 8 2023

Dynamical Systems and Fractals - Problem Set

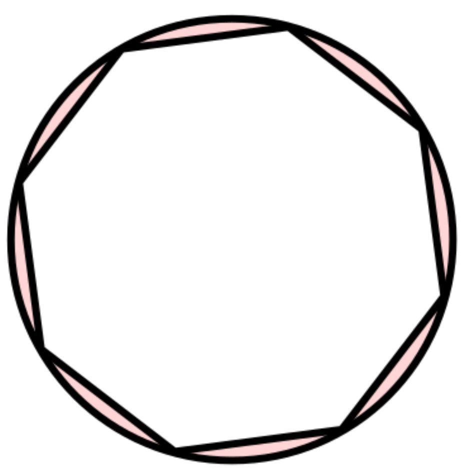
1. Consider the logistic function $f(x) = rx(1-x)$ where $0 < r \leq 4$. In today's lesson we saw numerically that when $r > 3$ this function has a two-cycle. Let's show this algebraically. We can solve for the period two points of $f(x)$ by solving the expression $f^{[2]}(\bar{x}) = \bar{x}$, however as $f(x)$ gets more complicated this can leave us with some messy equations to solve. In this question we will work through an easier way to solve for the two-cycle of $f(x)$.

   (a) Let $\{p_1, p_2\}$ be the two-cycle of $f(x)$. In order for this to be a two-cycle we must have that $f(p_1) = p_2$ and $f(p_2) = p_1$. Use this fact to write down two expressions relating p_1 and p_2.

   (b) Now subtract the two expressions you found in (a) and use the fact that $p_1 \neq p_2$ to simplify the resulting expression. You should end up with an expression which is linear in both p_1 and p_2.

   (c) Finally, substitute this expression back into one of the expressions you found in (a) to solve for either p_1 or p_2. Use this result to show that $f(x)$ only has a (real-valued) two-cycle when $r > 3$.

2. Let's say we have a circle C with radius 1. Now, consider inscribing C with a regular polygon P_n which has 2^n equal sides, as shown in the figure below (the figure shows P_3 since $2^3 = 8$ sides).



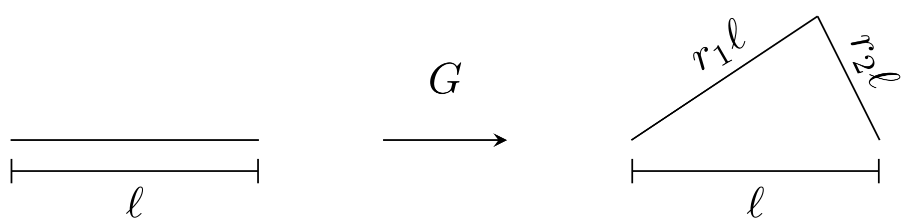
We can consider the length (L_n) of the perimeter of P_n as an approximation for the circumference ($L = 2\pi$) of C.

(a) Write down an expression for L_n (the length of the perimeter of P_n).

(b) **CHALLENGE** (You will need to be familiar with limits in order to solve this next part.) Show that $\lim_{n\to\infty} L_n = L = 2\pi$.

Hint: You may work with angles in either degrees or radians (if you are familiar with radians). You will need to use the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ *(when x is in radians) or that* $\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\pi}{180}$ *(when x is in degrees).*

3. Consider the generator G sketched below:



where $0 < r_1 < 1$, $0 < r_2 < 1$ and $1 < r_1 + r_2 < 2$.

(a) Starting with the set $J_0 = [0,1]$, sketch $J_1 = G(J_0)$ and $J_2 = G(J_1)$.

(b) What is the length (L_1) of J_1? What is the length (L_2) of J_2? In general, can you find an expression for the length (L_n) of $J_n = G^n(J_0)$?

(c) What do you expect to happen to the length of J_n as n gets infinitely large (i.e. as the set J_n approaches the attractor)?

4. Consider the following two function iterated function system (IFS) on $[0,1]$,

$$f_1(x) = \frac{1}{5}x, \quad f_2(x) = \frac{1}{5}x + \frac{4}{5}.$$

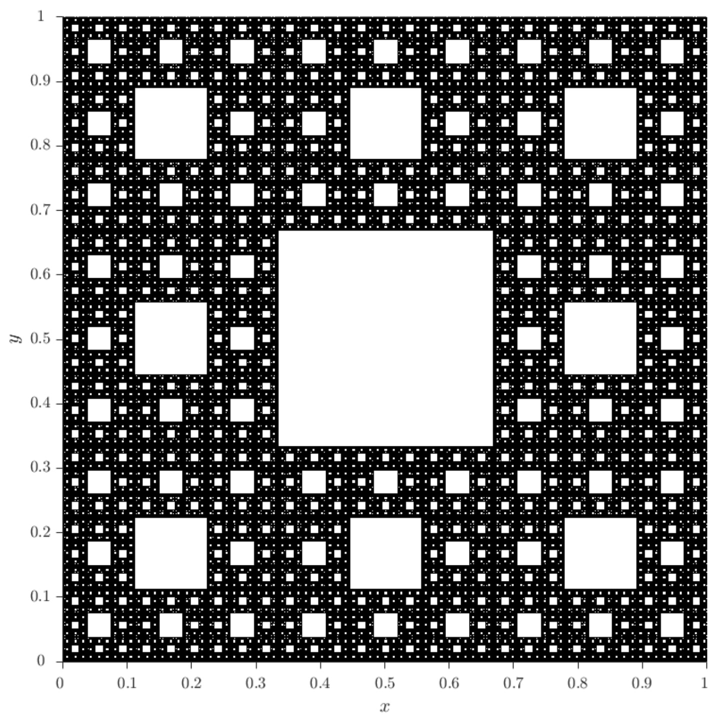
(a) Let $I_0 = [0,1]$ and $I_1 = F(I_0)$ where F is the parallel IFS operator composed of the two functions f_1 and f_2. Sketch I_1 on the real number line.

(b) Let $I_2 = F(I_1)$. Sketch I_2 on the real number line.

(c) Let I denote the limiting set (or attractor) of this IFS. Use the scaling relation to determine the fractal dimension D of I.

5. Show that the function $f(x) = x^2$ is a contraction mapping on the domain $D = [0, \frac{1}{4}]$. Determine the contraction factor of $f(x)$.

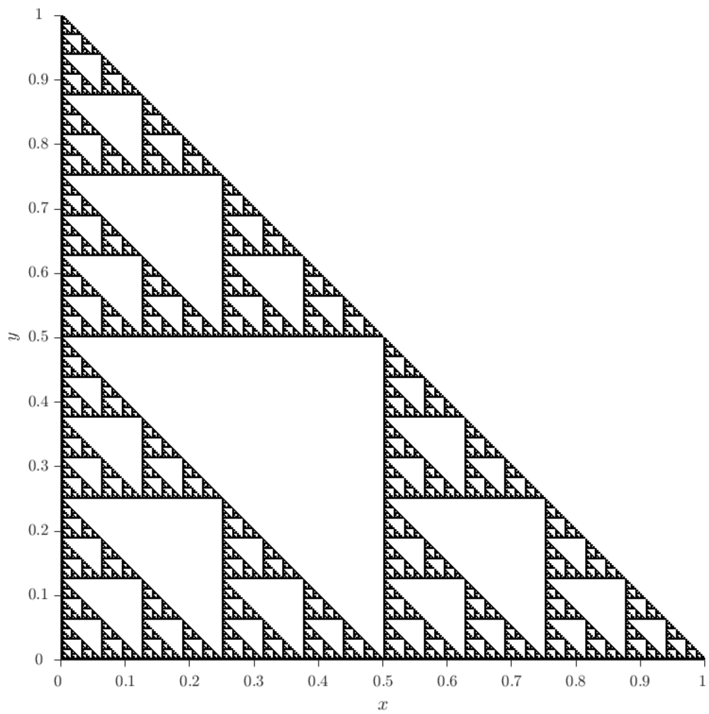
6. Consider the image of the Sierpinski carpet, S, shown below. The Sierpinski carpet is a self-similar fractal which means that is a union of contracted copies of itself.



   (a) Show (by circling them on the figure) that S is made up of eight contracted copies of itself. What is the contraction factor of these copies?

   (b) Determine the similarity dimension of S.

7. Consider the image of the modified Sierpinski triangle, S, shown below.



(a) Show (by circling them on the figure) that S is made up of three contracted copies of itself.

(b) Imagine starting with a right triangle, S_0, which has vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. Describe (in terms of contraction factors, translations, rotations, etc...) the three map IFS which you could use to construct S from S_0.

(c) Determine the similarity dimension of S.

(d) **CHALLENGE** Describe a fourth map which could be added to the IFS you found in (b) so that the attractor of the IFS is a solid triangular region.